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Wall-crossing from supersymmetric galaxies

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ABSTRACT: We give an elementary physical derivation of the Kontsevich-Soibelman wall crossing formula, valid for any theory with a 4d $\mathcal{N} = 2$ supergravity description. Our argument leads to a slight generalization of the formula, which relates monodromy to the BPS spectrum.

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1. Introduction

In [11] Kontsevich and Soibelman proposed a remarkable wall crossing formula for BPS indices. In this note we show that this formula can be derived in an elementary way from the halo picture of BPS bound states in supergravity [4, 5].

The basic strategy we follow is similar to that of [9], which gave a proof of the (motivic) KS wall-crossing formula in the context of $\mathcal{N} = 2$ *field theory*. The essential physical idea used halo configurations of particles bound to line operators. Our analysis will generalize this idea to gravity, without introducing external objects such as line operators. The surrogate for the line operator will be an infinitely massive BPS black hole, to which the BPS objects of interest are bound. The physical cartoon to have in mind is that of a galaxy with a supermassive black hole at its center, where the BPS objects of interest are the solar systems orbiting around it. These galactic configurations exhibit jumping phenomena when dialing the moduli: when crossing certain walls, halos of objects of a particular charge can be pushed out to infinity or conversely come in from infinity. The generating function for the BPS indices of these galactic bound states transforms in a simple way when such a wall is crossed, by the action of a certain operator on the generating function, which follows directly from the simple halo wall crossing formula (a.k.a. the semiprimitive wall crossing formula) of [5]. Collections of walls intersect on real codimension two loci, together also with marginal stability walls for the individual solar systems. Circling around these intersection loci will produce a sequence of wall crossing operations on the generating function. For a contractible loop in moduli space, the product of these operators must be the identity. This turns out to be nothing but the KS formula. For a noncontractible loop in moduli space we find a generalization of the KS formula.

We refer to the companion paper [1] for further background and examples. Our notation follows [1] which in turn follows the notation of [5].

2. BPS galaxies and the halo wall crossing operator

A halo is a BPS configuration consisting of an arbitrary number N of particles with electromagnetic charge proportional to a primitive charge γ surrounding a core of charge Γ . For simplicity of exposition (only!) we will initially consider only halo particles of charge γ . The charges are valued in a symplectic lattice L . The equilibrium distance R between core and halo particles is given by [3]

$$R = \frac{\langle \gamma, \Gamma \rangle}{2 \operatorname{Im}(e^{-i\alpha} Z_\gamma)}, \quad (2.1)$$

where $\langle \gamma, \Gamma \rangle$ is the electric-magnetic symplectic product of γ and Γ , Z_γ is the central charge of γ , measured at spatial infinity (where the vector multiplet moduli are set at $t = t_\infty$), and $\alpha = \arg Z_{\Gamma+N\gamma}$. A necessary condition for existence is $R > 0$. When the phases of the central charges of the core and halo line up, i.e. $\arg Z_\Gamma = \arg Z_\gamma = \alpha$, the radius diverges and the halo decays. Both core and halo particles can in turn be composites. The above formula for the equilibrium distance still holds as long as R is much larger than the size of these composites.

In the limit $R \rightarrow \infty$, the halo particles can be considered to be noninteracting electric point particles, confined to a sphere threaded by a uniform magnetic flux. The supersymmetric one particle ground states are given by the lowest Landau levels, and the N -particle halo states are constructed from those as an N particle Fock space $\mathcal{F}_\Gamma(N\gamma)$ [4, 5]. We denote the Witten index of these halo states by¹

$$\Omega_\Gamma^{\text{Fock}}(N\gamma; t_\infty) \equiv \operatorname{Tr}_{\mathcal{F}_\Gamma(N\gamma)}(-1)^F. \quad (2.2)$$

For $N = 1$, we have $\Omega_\Gamma^{\text{Fock}}(\gamma) = |\langle \gamma, \Gamma \rangle| \Omega(\gamma)$. Here $\Omega(\gamma)$ is the usual $\mathcal{N} = 2$ BPS index, and $|\langle \gamma, \Gamma \rangle|$ is the lowest Landau level degeneracy factor. For general N it is convenient to define a generating function. Introduce formal variables X_i , $i = 1, \dots, \operatorname{rank} L$, and write $X^\Delta := \prod_i X_i^{\Delta_i}$ for a charge Δ with components Δ_i with respect to some chosen basis for L . Then the generating function is

$$G_\Gamma^{\text{Fock}}(X) := \sum_N \Omega_\Gamma^{\text{Fock}}(N\gamma) X^{\Gamma+N\gamma} = \left(1 - (-1)^{\langle \gamma, \Gamma \rangle} X^\gamma\right)^{\Omega(\gamma)|\langle \gamma, \Gamma \rangle|} X^\Gamma \quad (2.3)$$

This follows from standard Fock space combinatorics [5].

In general $\Omega(\Gamma + N\gamma) \neq \Omega(\Gamma) \Omega_\Gamma^{\text{Fock}}(N\gamma)$ in the full theory. The reason is that the true index $\Omega(\Gamma + N\gamma)$ in general gets contributions from many other configurations of charges summing up to the same total charge. For instance a core black hole of charge Γ with two halo particles of charge γ and a core black hole of charge $\Gamma + \gamma$ and one halo particle of charge γ will both contribute to $\Omega(\Gamma + 2\gamma)$. At finite R , the corresponding Fock spaces can be expected to get mixed due to quantum tunneling between these configurations. Only the sum over all possible configurations is guaranteed to give a well defined index. Phrased differently, whereas the supersymmetric quantum

¹The indices depend on the background moduli t_∞ . For notational compactness we will sometimes suppress this dependence.

mechanics of halo particles trapped in their potential minimum at finite R is a well-defined closed system in perturbation theory, nonperturbative tunneling between this minimum and the minimum corresponding to merging with the black hole core causes the wave function of the halo configuration to “leak out” and mix with configurations with different core black hole charges. It is no longer a well-defined closed system.

The leaking can be prevented, however, by taking the limit of infinite core black hole size, as black hole tunneling is generically exponentially suppressed in the size of the black hole. This is entirely an entropic effect. For example the amplitude for fragmentation of an extremal Reissner-Nordström black hole of charge $Q = Q_1 + Q_2$ into black holes of charge Q_1 and Q_2 — a process unobstructed by any potential barrier — is nevertheless suppressed as $e^{-\frac{1}{2}\Delta S}$ where $\Delta S = \pi Q^2 - \pi Q_1^2 - \pi Q_2^2 = 2\pi Q_1 Q_2$ [13]. Therefore in the $Q \rightarrow \infty$ limit, taking into account charge quantization, the extremal RN black hole becomes absolutely stable; there is no more mixing with fragmented configurations. Stability of large black holes is a universal phenomenon — even Schwarzschild black holes stop radiating and become stable in the infinite size limit.

Thus, we will consider configurations of BPS objects orbiting around a supermassive black hole core of charge Γ_c , where we eventually send $\Gamma_c \rightarrow \infty$ while keeping the total charge of the objects in the orbits finite. The objects themselves can be multicentered BPS bound states. We can loosely think of this system as a galaxy consisting of many solar systems orbiting around a supermassive black hole, and we therefore refer to these objects as “BPS galaxies”. The simplest situation is when we have a single halo of particles of charge γ around the hole, but we also allow multiple halos, or more general, non-halo configurations involving interacting solar systems with mutually nonlocal charges. So the most general BPS galaxy will be a complex multi-particle bound state, with potentially strong position-constraining interactions between neighboring solar systems, and intricate exchanges of suns and planets between different solar systems possible when dialing the moduli.

To make this more precise, we have to specify more carefully how we take the limit $\Gamma_c \rightarrow \infty$. For our purpose of deriving the KS formula, it turns out to be convenient to single out a particular $U(1)$, give the core large electric and magnetic charges with respect to this $U(1)$, and keep the orbiting solar systems uncharged under this $U(1)$. More precisely, we choose a set of charges $C \equiv \{\Gamma_0, \Gamma'_0, \gamma_c\}$ such that Γ_0 supports a single centered BPS black hole, $\langle \Gamma_0, \Gamma'_0 \rangle \neq 0$, and $\langle \gamma_c, \Gamma_0 \rangle = 0 = \langle \gamma_c, \Gamma'_0 \rangle$. We then set

$$\Gamma_c = \Lambda^2 \Gamma_0 + \Lambda \Gamma'_0 + \gamma_c \quad (2.4)$$

and take $\Lambda \rightarrow \infty$. The anisotropic scaling is chosen for reasons that will become clear later (see footnote 7). To avoid infinite lowest Landau level degeneracies, we restrict the charges γ of the solar systems orbiting around this core to be orthogonal to both Γ_0 and Γ'_0 , which means they are uncharged under the $U(1)$ associated to Γ_0 and Γ'_0 . More formally, the sublattice of orbiting charges

L_{orb} is thus

$$L_{\text{orb}} := \{\gamma \in L \mid \langle \gamma, \Gamma_0 \rangle = 0 = \langle \gamma, \Gamma'_0 \rangle\}. \quad (2.5)$$

With this definition, we also have $\gamma_c \in L_{\text{orb}}$.

The Hilbert space of BPS galaxies with core charge Γ_c and total orbital charge Γ_{orb} has an overall factor corresponding to the internal states of the core black hole, which we can factor out to produce a factor space $\mathcal{H}_{\Gamma_c}(\Gamma_{\text{orb}}; t_\infty)$, which can be thought of as the Hilbert space of the orbiting solar systems in a background sourced by the core black hole. We obtain a closed supersymmetric quantum system with this Hilbert space provided there is no mixing between galaxies of different core charges, nor mixing with galaxies which do contain charges in orbit which are *not* in the restricted lattice L_{orb} . This turns out to be generically the case in the limit $\Lambda \rightarrow \infty$, essentially because such tunneling events are either infinitely entropically suppressed along the lines mentioned above, or infinitely suppressed because they require tunneling over infinite distances. We give detailed arguments for this in appendix A, and prove that there is just one exception, namely when it so happens that the attractor point of Γ_0 lies on a locus with massless particles with charge in L_{orb} , in which case there may be mixing between galaxies with cores differing by the charges becoming massless. This situation is nongeneric, and for the remainder of the paper we will assume this is not the case.

Thus, at fixed Γ_{orb} , in the limit $\Lambda \rightarrow \infty$, we can define a proper Witten index for this supersymmetric closed system, which we call the “framed” BPS galaxy index, in analogy with the framed BPS indices of [9]:

$$\overline{\Omega}_C(\Gamma_{\text{orb}}; t_\infty) := \lim_{\Lambda \rightarrow \infty} \text{Tr}_{\mathcal{H}_{\Gamma_c}(\Gamma_{\text{orb}}; t_\infty)} (-1)^F. \quad (2.6)$$

Here $C \equiv \{\Gamma_0, \Gamma'_0, \gamma_c\}$ is the set of charges determining the one parameter family $\Gamma_c(\Lambda)$ of core charges as in (2.4). It will be useful to introduce the generating function of framed BPS indices:

$$G_C(X; t_\infty) := \sum_{\Gamma_{\text{orb}} \in L_{\text{orb}}} \overline{\Omega}_C(\Gamma_{\text{orb}}; t_\infty) X^{\gamma_c + \Gamma_{\text{orb}}}. \quad (2.7)$$

The presence of singularities and associated monodromies gives rise to some subtleties, which we discuss in section 4. For the time being we simply assume we stay in a sufficiently small open set of moduli space, away from singular loci, in which case we can ignore these subtleties altogether.

The key observation that makes this construction useful is that although the generic BPS galaxy has a very complicated structure, its wall crossing behavior when varying t_∞ is very simple. It is entirely governed by pure halo decays, since the galactic core black hole cannot decay and serves as a fixed, primitively charged center. Whenever the central charge $Z(\gamma)$ of some charge γ supporting BPS states lines up with the total central charge $Z = Z(\Gamma_c) + Z(\Gamma_{\text{orb}})$ of the galaxy, a halo of objects with charge γ can be added or subtracted at spatial infinity. We again restrict to $\gamma \in L_{\text{orb}}$. In the $\Lambda \rightarrow \infty$ limit the wall in moduli space where this happens is independent of the solar system charge, since

in this limit $Z/Z(\Gamma_c) = 1$, so $\arg Z = \arg Z(\Gamma_c) = \arg Z(\Gamma_0)$ and we can set $\alpha = \alpha_0 := \arg Z(\Gamma_0; t_\infty)$ in (2.1). Hence the wall of marginal stability for the halo is ²

$$W_\gamma = \{t_\infty | \arg[e^{-i\alpha_0} Z(\gamma, t_\infty)] = 0\}, \quad \text{stable side: } \langle \gamma, \gamma_c + \Gamma_{\text{orb}} \rangle \text{Im}[e^{-i\alpha_0} Z(\gamma, t_\infty)] > 0. \quad (2.8)$$

We will call these “BPS walls.”

The part of the Hilbert space of all BPS galaxies with fixed core charge Γ_c that jumps across a BPS wall W_γ is given by the halo Fock space described earlier, with an effective core charge Γ , as seen by this halo, given by the *total* interior galactic charge $\Gamma = \Gamma_c + \Gamma_{\text{orb}}$ enclosed by the halo. The corresponding transformation of the framed galactic indices can therefore be inferred from (2.3). Roughly speaking, the terms in the generating function G_C in (2.7) get multiplied by the factor appearing in (2.3). However, as we have just explained, the effective Γ appearing in (2.3) depends on Γ_{orb} and hence is different for the different terms in G_C , and so the multiplication factor will be different. This is easily formalized by introducing a linear operator D_γ acting on monomials X^δ by pulling down the symplectic product:

$$D_\gamma X^\delta := \langle \gamma, \delta \rangle X^\delta. \quad (2.9)$$

With this and an eye on (2.3), we define the following operator acting on polynomials in X : ³

$$T_\gamma := (1 - (-1)^{D_\gamma} X^\gamma)^{D_\gamma}. \quad (2.10)$$

Notice that this operator effectively acts as a diffeomorphism on the coordinates X^i . The transformation of the generating function when crossing the wall W_γ in the direction of *increasing* $\arg[Z_\gamma e^{-i\alpha_0}]$ is then

$$G_C(X) \rightarrow U_\gamma(t) G_C(X), \quad U_\gamma(t) := \prod_{k \in \mathbb{Z}^+} T_{k\gamma}^{\Omega(k\gamma; t)}, \quad (2.11)$$

where we made the dependence on the point t where the wall is crossed explicit. We take γ to be primitive. The product over k comes from the fact that the walls $W_{k\gamma}$ coincide. (Thus, we have now relaxed our initial assumption that only halo particles of primitive charge γ enter.) To check that this formula is correct when going in the direction of increasing $\arg[Z_\gamma e^{-i\alpha_0}]$, note that on the part of the generating function for which $D_\gamma > 0$, going in this direction means by (2.8) going from the unstable to the stable side, and vice versa for the $D_\gamma < 0$ part. Therefore, the wall crossing formula should multiply the $D_\gamma > 0$ terms by halo factors (2.3), and conversely remove such factors from the $D_\gamma < 0$ part (or alternatively add such factors when the inverse operation is performed,

²These are analogs of the “BPS walls” of [9] with $e^{i\alpha_0}$ playing the role of ζ . However, an important difference is that now $e^{i\alpha_0}$ depends on t_∞ and is only an independent variable to the extent that Γ_0 is.

³We remark that the operators $\tau_\gamma := (-1)^{D_\gamma} X^\gamma$ satisfy $\tau_\gamma \tau_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \tau_{\gamma+\gamma'}$, and hence the operators τ_γ provide a natural quadratic refinement of the mod-two intersection form, a point which aficionados of the KSWCF will surely appreciate. (A related point was made in equations (3.27)-(3.29) of [9].)

corresponding to decreasing $\arg[Z_\gamma e^{-i\alpha_0}]$. This is indeed implemented by the fact that we dropped in (2.11) the absolute value signs appearing in the exponent of (2.3).

Finally, we come to the central formula of this Letter. Consider a closed contractible loop \mathcal{P} in moduli space (noncontractible loops will be discussed in section 4). Along this loop, the generating function G_C will undergo a sequence of wall crossing operations $U_{\gamma_i}(t_i)$. Since \mathcal{P} is contractible, the composition of these operations must act trivially on G_C , for any choice of γ_c and starting point t :

$$\prod_i U_{\gamma_i}(t_i) \cdot G_C = G_C, \quad (2.12)$$

where the product is ordered according to the sequence of walls crossed: points crossed later in the path are placed to the left. At the core attractor point $t_*(\Gamma_c)$ there are no multicentered bound states involving Γ_c , and hence no BPS galaxies. So at this point we have simply

$$G_C(X)|_{t_*(\Gamma_c)} = X^{\gamma_c}. \quad (2.13)$$

Starting from this expression, the wall crossing formula (2.11) uniquely determines all framed galactic indices given all $\Omega(k\gamma)$. This shows that G_C is well defined as a function to the extent that the wall crossing factors are. (It is conceivable that a dense set of BPS walls can lead to an ill-defined expression.) Furthermore by varying γ_c we can generate as many independent functions $G_C(X)$ as there are independent variables X_k associated to charges in L_{orb} .⁴ This in combination with the fact that the wall crossing operators U_γ act as diffeomorphisms implies that the product of the sequence of halo wall crossing operators around a contractible loop must be the identity

$$\prod_i U_{\gamma_i}(t_i) = 1. \quad (2.14)$$

We will prove in detail in the next section that this is in essence equivalent to the KS wall crossing formula.

3. Derivation of the KS formula

We now demonstrate that when \mathcal{P} is a small contractible loop intersecting a wall of marginal stability the Kontsevich-Soibelman wall crossing formula is a consequence of (2.14). Let us therefore consider two mutually nonlocal charges γ_1, γ_2 and a generic *non-singular* point $t_{ms} \in MS(\gamma_1, \gamma_2)$ where both central charges are nonzero and γ_1, γ_2 support BPS states. Using the attractor equation it is easy to show that we can always find a Γ_0 (and hence a phase α_0) so that Γ_0 supports single-centered black

⁴This corresponds to the condition, discussed in [9], that there are “enough” line operators to deduce the KSWCF.

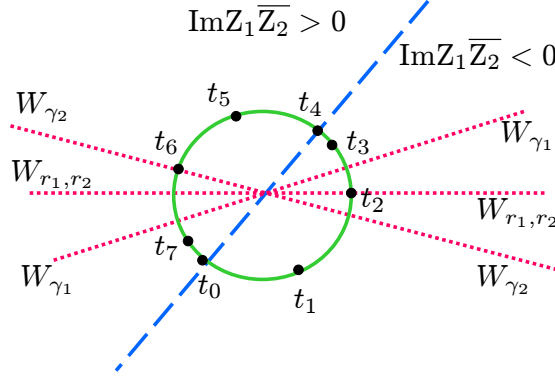


Figure 1: This shows the neighborhood \mathcal{U} in the normal bundle to $W_{\gamma_1} \cap W_{\gamma_2}$. The wall of marginal stability is given by $\text{Im}[Z(\gamma_1; t)\overline{Z(\gamma_2; t)}] = 0$ since $\text{Re}[Z(\gamma_1; t)\overline{Z(\gamma_2; t)}]$ is nonzero throughout \mathcal{U} . We choose the ordering of γ_1, γ_2 so that W_{γ_1} is counterclockwise from W_{γ_2} with opening angle smaller than π . Then the BPS walls $W_{r_1\gamma_1+r_2\gamma_2}$ are ordered so that increasing r_1/r_2 gives walls in the counterclockwise direction. We consider a path \mathcal{P} in \mathcal{U} circling the origin in the counterclockwise direction. The central charges of vectors $r_1\gamma_1 + r_2\gamma_2$ with $r_1, r_2 \geq 0$ at representative points t_0, \dots, t_7 along \mathcal{P} are illustrated in the next figure.

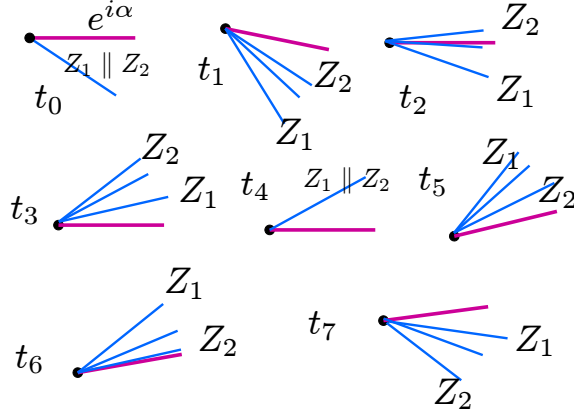


Figure 2: As t moves along the path \mathcal{P} the central charges evolve as in this figure. Note that $\text{Im}(Z_1\overline{Z_2}) > 0$ means that Z_1 is counterclockwise to Z_2 and rotated by a phase less than π . In that case the rays parallel to $r_1Z_2 + r_2Z_1$ for $r_1, r_2 \geq 0$ are contained in the cone bounded by $Z_1\mathbb{R}_+$ and $Z_2\mathbb{R}_+$, and ordered so that increasing r_1/r_2 corresponds to moving counterclockwise. When t crosses the marginal stability wall the cone collapses and the rays reverse order. As t moves in the region t_2 the quantity $\arg[Z_\gamma e^{-i\alpha_0}] > 0$ is increasing for all γ_{r_1, r_2} with $r_1, r_2 \geq 0$ while at the point t_6 the argument is decreasing.

holes and t_{ms} lies on the intersection of BPS walls $W_{\gamma_1} \cap W_{\gamma_2}$.⁵ This intersection is real codimension

⁵We can take for example $\Gamma_0 \equiv -2\text{Im}[\bar{X}\Omega^{(3,0)}]$, where $\Omega^{(3,0)}$ is the holomorphic 3-form evaluated at t_{ms} and X

two in moduli space and we now consider a small neighborhood \mathcal{U} of t_{ms} so that the only other BPS walls $W_{\gamma'}$ passing through t_{ms} arise from charges of the form $\gamma' = r_1\gamma_1 + r_2\gamma_2$ for rational r_1, r_2 . We will denote charges of this form by γ_{r_1, r_2} . Since the point t_{ms} is non-singular a loop around it is contractible and (2.14) holds.

Below we will argue that, perhaps after choosing suitable linear combinations, we can assume that the only populated charges of type γ_{r_1, r_2} in \mathcal{U} in fact have $(r_1, r_2) \in \mathbb{Z}^2$ with r_1, r_2 both ≥ 0 or both ≤ 0 . We can order γ_1, γ_2 so that the configuration of BPS walls and the marginal stability wall are arranged as shown in Figure 1. Suppose we begin at the point t_1 and move along the path \mathcal{P} in the counterclockwise direction. We first cross the BPS walls in the region $\text{Im}Z_1\bar{Z}_2 < 0$ in order of increasing r_1/r_2 and *increasing* $\arg[Z_{\gamma_{r_1, r_2}} e^{-i\alpha_0}]$. Then we cross in the region $\text{Im}Z_1\bar{Z}_2 > 0$ again with increasing r_1/r_2 but now this corresponds to *decreasing* values of $\arg[Z_{\gamma_{r_1, r_2}} e^{-i\alpha_0}]$. Thus we have

$$\prod_{\substack{\leftarrow \\ \frac{r_1}{r_2} \nearrow}} T_{r_1, r_2}^{-\Omega_{r_1, r_2}^+} \prod_{\substack{\leftarrow \\ \frac{r_1}{r_2} \nearrow}} T_{r_1, r_2}^{\Omega_{r_1, r_2}^-} = 1 \quad (3.1)$$

where the arrows on the product mean that increasing values of r_1/r_2 are written to the left, and Ω_{r_1, r_2}^\pm is the BPS index of $r_1\gamma_1 + r_2\gamma_2$ in the region \mathcal{U} with $\text{Im}Z_1\bar{Z}_2 > 0$ and < 0 respectively. Taking into account the relation between the ordering of r_1/r_2 and the ordering of the phases of the central charges illustrated in figures 1 and 2 we can also write this in the more traditional way:

$$\prod_{\substack{\rightarrow \\ \arg Z_{r_1, r_2} \nearrow}} T_{r_1, r_2}^{\Omega_{r_1, r_2}^+} = \prod_{\substack{\rightarrow \\ \arg Z_{r_1, r_2} \nearrow}} T_{r_1, r_2}^{\Omega_{r_1, r_2}^-}. \quad (3.2)$$

This is the KS wall crossing formula.

We still need to fill in a gap above and justify the important claim that we can choose γ_1, γ_2 so that only r_1, r_2 both ≥ 0 or ≤ 0 are populated. This “root basis property” can be rigorously proven in certain field theory examples [8]. We offer an alternative justification here by requiring that the spectrum of BPS masses should not have an accumulation point at zero. (We are therefore using that the point t_{ms} is not at a singularity of moduli space since that assumption is violated, for example at a superconformal point.) Denoting the central charges of γ_1, γ_2 at t_{ms} by ρ_1, ρ_2 we therefore know that there is an $\epsilon > 0$ so that populated charges γ_{r_1, r_2} must have $|r_1\rho_1 + r_2\rho_2| > \epsilon$. In the (r_1, r_2) -plane this is a strip of width 2ϵ centered on the line with slope $-\rho_2/\rho_1$. (Since t_{ms} is on the marginal stability wall ρ_2/ρ_1 is real.) If our point t_{ms} is generic then there is in fact a neighborhood of t_{ms} in the marginal stability wall so that, moving along this wall the spectrum of

is an arbitrary complex constant with $\arg X \equiv \arg Z_1 = \arg Z_2$. This Γ_0 has a regular attractor point, namely t_{ms} , because the equation we used to define Γ_0 is nothing but the attractor point equation. Taking the symplectic product of this equation with γ_1, γ_2 shows that $\langle \gamma_1, \Gamma_0 \rangle = 0 = \langle \gamma_2, \Gamma_0 \rangle$. Taking the symplectic product with $\Omega^{(3,0)}$ shows that $X = Z(\Gamma_0 : t_{ms})$, so, as we wished, the central charges line up at t_{ms} . Although Γ_0 will in general not be quantized, this is acceptable since all we care about in the end is the limit $\Lambda \rightarrow \infty$.

BPS particles of charges of the form γ_{r_1, r_2} must remain constant. But the slope $-\rho_2/\rho_1$ will vary in this neighborhood. This means that there must be an unpopulated wedge (and its negative) in the (r_1, r_2) -plane. By choosing a suitable redefinition $\gamma_1 \rightarrow a\gamma_1 + b\gamma_2, \gamma_2 \rightarrow c\gamma_1 + d\gamma_2$ we can ensure that the populated states in the complementary wedges are of the form γ_{r_1, r_2} with r_1, r_2 both ≥ 0 or both ≤ 0 .

We end with two remarks

1. The root basis property of BPS states is addressed in the mathematical framework of Kontsevich and Soibelman [11] in a slightly different way. A part of their “stability conditions,” used a quadratic form on the lattice of charges $\mathcal{Q} : L \rightarrow \mathbb{R}$ and only the charges that satisfied $\mathcal{Q}(\gamma) \geq 0$ were considered. The quadratic form also has the property that $\mathcal{Q}|_{\text{Ker } Z} < 0$, where Z is the central charge map $Z : L \rightarrow \mathbb{C}$. Thus, restricting the set of charges entering the WCF to $\mathcal{Q}(\gamma) \geq 0$ means that we have to discard certain wedges in the space of charges surrounding the directions with $Z(\gamma) = 0$.
2. Finally, we comment on the “motivic” or “refined” version of the wall-crossing formula [11] which takes into account spin degrees of freedom [6, 7]. The field theoretic derivation of the motivic KSWCF given in [9] can also be carried over directly in the present context: We now let X_γ be valued in the quantum torus. We replace G_C by the generating function of the spin characters, and across the walls W_γ we will find that G_C is conjugated by certain combinations of quantum dilogarithms. However, we stress that the justification for the derivation in [9] relied on the existence of “*protected spin characters*,” which can only be defined if there is an $SU(2)_R$ symmetry in the supersymmetry algebra. In general this symmetry is not present in supergravity, and hence, from the viewpoint of this Letter, the validity of “motivic” generalization of the wall-crossing formula is a little mysterious. In fact, as is well-known, the spin character depends on hypermultiplet moduli as well as vectormultiplet moduli. (For examples in the weakly coupled heterotic strings with type II duals see [10, 2].)

4. Generalization to noncontractible loops

In our derivation of the KS formula, we considered a contractible loop \mathcal{P} in moduli space. Nothing prevents us from considering instead a non-contractible loop, in particular a loop circling around a point on the discriminant locus. Such a loop will be closed in moduli space but not in covering space, and the local system of charges undergoes nontrivial monodromy $M_{\mathcal{P}} : L \rightarrow L$ after going around it.⁶ As a result the generating function will not be exactly preserved, and (2.12) must be modified.

⁶To avoid cluttering the discussion, in the following we will not bother specifying at each step in which direction we orient loops, monodromies etc.

As mentioned under (2.7), the proximity of singularities associated to nontrivial monodromies can lead to some subtleties in the definition of the framed BPS indices $\overline{\underline{\Omega}}_C(\Gamma_{\text{orb}}, t_\infty)$. Besides the usual jumps at marginal stability, there are two other kinds of formal index “jumps” (or rather relabelings) related to the presence of singularities and monodromies. The first occurs when t_∞ crosses a cut, where the choice of charge lattice basis jumps by convention. This is just a relabeling of indices, equating framed indices involving charges related by the corresponding basis transformation. If desired it can be eliminated by going to the moduli covering space. The second event occurs when t_∞ crosses a “conjugation wall” in the language of [1], i.e. when the core attractor flow gets “pulled through” a singular locus in moduli space. In this case new particles (becoming massless at the singularity) appear in orbit while the apparent core charge as seen from infinity jumps, keeping the total charge (and index) unchanged. This is again some kind of relabeling of indices, equating framed indices involving shifted core and orbit charges, but this time the jump cannot be eliminated by going to the covering space.

More formally, when crossing a cut from t_∞ to t'_∞ , charges $\Gamma|_{t'_\infty}$ and $M \cdot \Gamma|_{t_\infty}$ get identified. Thus the indices on the respective sides of the cut are related by

$$\overline{\underline{\Omega}}_C(\Gamma_{\text{orb}}; t'_\infty) = \overline{\underline{\Omega}}_{M \cdot C}(M \cdot \Gamma_{\text{orb}}; t_\infty). \quad (4.1)$$

A short computation shows that the generating functions get accordingly identified as

$$G_C(X; t_\infty) = \hat{M} \cdot G_{M^{-1}C}(X, t'_\infty), \quad (4.2)$$

where we defined for any automorphism M of the charge lattice a map on generating functions by

$$\hat{M} \cdot \sum_{\Gamma} a_{\Gamma} X^{\Gamma} := \sum_{\Gamma} a_{\Gamma} X^{M \cdot \Gamma}. \quad (4.3)$$

When crossing a conjugation wall from t_∞ to t'_∞ , by definition, the core attractor flow gets pulled through the discriminant locus, so that if initially the core attractor flow did not cross the cut ending on the discriminant locus, it now does, or vice versa. By physical continuity, the core charge as seen by a local observer at the core must remain Γ_c . Hence, if the monodromy transformation associated to the cut is $\Gamma \rightarrow M \cdot \Gamma$, the apparent core charge as seen by an observer at spatial infinity jumps from Γ_c to $M \cdot \Gamma_c$. Since the total charge must remain the same, the charge in the galactic orbit must jump from Γ_{orb} to $\Gamma_{\text{orb}} + (1 - M) \cdot \Gamma_c$ (see [1] for a detailed discussion of how this happens physically). Note that to remain in the picture in which the orbit charge remains finite when $\Lambda \rightarrow \infty$, we should therefore require

$$M \cdot \Gamma_0 = \Gamma_0, \quad M \cdot \Gamma'_0 = \Gamma'_0. \quad (4.4)$$

The framed indices on the respective sides of the conjugation wall are then related by

$$\overline{\underline{\Omega}}_C(\Gamma_{\text{orb}}; t_\infty) = \overline{\underline{\Omega}}_{M \cdot C}(\Gamma_{\text{orb}} + (1 - M) \cdot \gamma_c; t'_\infty). \quad (4.5)$$

The corresponding generating functions are related even more simply by

$$G_C(X, t_\infty) = G_{M \cdot C}(X, t'_\infty). \quad (4.6)$$

We can now collect these results and state the generalization of (2.12) to the case of a noncontractible loop \mathcal{P} around a point t_0 of the discriminant locus, with associated monodromy M . As before, we assume that no massless BPS particles exist at $t_*(\Gamma_0)$. Since in general there are massless BPS particles present at the discriminant locus, we assume in particular that we have chosen Γ_0 to be such that $t_0 \neq t_*(\Gamma_0)$. There are two cases to distinguish:

1. **Singularity without conjugation wall:** This is the case for singularities at infinite distance, such as the infinite volume limit of IIA on the quintic. We can assume there is a single cut ending on the singularity, across which the generating function transforms as in (4.2). Going infinitesimally across the cut in one direction or along the full loop \mathcal{P} in the other direction (along which the generating function undergoes a series of wall crossing operations as before), should give the same result. Thus (2.12) generalizes to

$$\prod_i U_{\gamma_i}(t_i) \cdot G_C = \hat{M} \cdot G_{M^{-1} \cdot C}. \quad (4.7)$$

2. **Singularity with conjugation wall:** This is the case typically for singularities at finite distance, such as the conifold point of IIA on the quintic. Assuming (4.4) and taking without loss of generality the cut on top of the conjugation wall for convenience, the transformation of the generating function when crossing the wall is given simply by $G_C \rightarrow G_{M \cdot C} \rightarrow \hat{M} \cdot G_C$, and the analog of (2.12) becomes

$$\prod_i U_{\gamma_i}(t_i) \cdot G_C = \hat{M} \cdot G_C. \quad (4.8)$$

By the same arguments as before, we can infer from this the operator equation

$$\prod_i U_{\gamma_i}(t_i) = \hat{M}, \quad (4.9)$$

which generalizes (2.14).

As an application of this formula, consider a singularity t_0 where a charge γ becomes massless, but no other linearly independent charges do. Because $Z(\gamma)$ acquires all phases around t_0 , the loop \mathcal{P} will necessarily cross both W_γ and $W_{-\gamma}$. If the loop is chosen such that these are the only walls that are crossed, equation (4.9) becomes

$$\begin{aligned} \hat{M} &= U_{-\gamma} \cdot U_\gamma \\ &= \prod_k \left(1 - (-1)^{-kD_\gamma} X^{-k\gamma} \right)^{-k\Omega(k\gamma)D_\gamma} \prod_k \left(1 - (-1)^{kD_\gamma} X^{k\gamma} \right)^{k\Omega(k\gamma)D_\gamma} \\ &= X^{\sum_k k^2 \Omega(k\gamma) \gamma D_\gamma}. \end{aligned} \quad (4.10)$$

Recalling (4.3), we see this is equivalent to

$$M \cdot \Gamma = \Gamma + \sum_k k^2 \Omega(k\gamma) \langle \gamma, \Gamma \rangle \gamma. \quad (4.11)$$

Thus this generalized KS formula relates monodromy to the the BPS spectrum. In the case of the simple conifold, $\Omega(k\gamma) = \delta_{k,0}$ and the above formula reduces to the well know Picard-Lefschetz monodromy formula $M \cdot \Gamma = \Gamma + \langle \gamma, \Gamma \rangle \gamma$. We discuss such relations in much more detail in [1].

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A. No-mixing conditions

A crucial element in our derivation of the KS wall crossing formula and its generalization was the argument for the absence of quantum mixing between galaxies with different core charges, and between galaxies with orbit charges $\gamma \in L_{\text{orb}}$ and galaxies with some orbit charges $\gamma \notin L_{\text{orb}}$. As promised we will now examine this argument in more detail, and show that mixing is absent in the $\Lambda \rightarrow \infty$ limit except if there exist massless charged particles at the attractor point of Γ_0 , with charge in L_{orb} .

We first investigate nonperturbative quantum mixing between the perturbative semiclassical states corresponding to a galaxy with all orbiting charges $\gamma \in L_{\text{orb}}$, i.e. $\langle \gamma, \Gamma_0 \rangle = 0 = \langle \gamma, \Gamma'_0 \rangle$, and those corresponding to a galaxy with some orbiting charges $\gamma \notin L_{\text{orb}}$. The core charge is $\Gamma_c = \Lambda^2 \Gamma_0 + \Lambda \Gamma'_0 + \gamma_c$ for both galaxies. This kind of mixing could in principle be mediated by a tunneling process in which a charge γ in orbit splits into a charge $\gamma_1 + \delta$ and a charge $\gamma_2 - \delta$, with $\gamma_1, \gamma_2 \in L_{\text{orb}}$, $\gamma_1 + \gamma_2 = \gamma$, and $\delta \notin L_{\text{orb}}$, followed by tunneling of the two charges to their respective BPS equilibrium positions. If the charges are held fixed in the $\Lambda \rightarrow \infty$ limit, then since $\delta \notin L_{\text{orb}}$ the symplectic product $\langle \delta, \Gamma_c \rangle$ is at least of order Λ and therefore by (2.1) the distance to which the charges would have to tunnel diverges when $\Lambda \rightarrow \infty$. Since tunneling over infinite distances is infinitely suppressed, the amplitude for such a process vanishes in the limit. If on the other hand we allow δ to grow with Λ , then in particular for $\delta \equiv \Lambda \Gamma_0 + \Gamma'_0$, it is no longer true that $\langle \delta, \Gamma_c \rangle$ diverges. So for such δ the tunneling trajectory does not have to be infinitely long. However, such diverging charges carry diverging entropy, and hence, by the arguments we will give below, we get infinite entropic tunneling suppression of the splitting event. An even stronger argument is that

BPS configurations containing such charge pairs $(\gamma_1 + \delta, \gamma_2 - \delta)$ actually cannot exist, since in the limit $\Lambda \rightarrow \infty$, these two charges are essentially opposite (as they diverge but sum up to a finite fixed charge γ), so they are essentially each others antiparticles, and it is not possible to have particles and anti-particles at the same time in orbit and still be BPS (since particle annihilation would clearly be energetically favorable). Thus, either way, mixing with galaxies with orbiting charges not in L_{orb} does not occur in the limit $\Lambda \rightarrow \infty$.⁷

Now we investigate mixing between different cores. Consider a BPS galaxy with core charge $\Gamma_c = \Lambda^2 \Gamma_0 + \Lambda \Gamma'_0 + \gamma_c$ and total orbiting charge Γ_{orb} , and a galaxy with core charge $\Gamma'_c = \Gamma_c - \delta$ and orbiting charge $\Gamma'_{\text{orb}} = \Gamma_{\text{orb}} + \delta$. The perturbative semiclassical states corresponding to these classical configurations can mix nonperturbatively through tunneling of a BPS particle of charge δ between the core black hole and a solar system orbiting the galaxy. We will now argue that such tunneling is infinitely suppressed in the limit $\Lambda \rightarrow \infty$, except if δ lies in L_{orb} and becomes massless at the attractor point $t_*(\Gamma_0)$ of Γ_0 .

The infinite suppression when $\delta \notin L_{\text{orb}}$ in the limit $\Lambda \rightarrow \infty$ follows by essentially the same arguments as we used above to show the absence of mixing between galaxies with all orbiting charges in L_{orb} and galaxies with some orbiting charges not in L_{orb} : charges $\delta \notin L_{\text{orb}}$ would either have to tunnel infinitely far (when they are kept finite), or (when $\delta \propto \Lambda \Gamma_0 + \Gamma'_0$) have infinite entropy themselves and give rise to an infinite change in entropy of the core. Either way, tunneling is infinitely suppressed.

When $\delta \in L_{\text{orb}}$, the particle can tunnel to finite distance, but tunneling will be infinitely suppressed in the limit due to the fact that the change in entropy of the core is infinite, *except* when the mass $m_\delta = |Z_\delta|$ of δ vanishes at $t_*(\Gamma_0)$. We first show the steps in the proof of this claim and then explain them. The entropy difference is

$$\Delta S = S_{\text{BH}}(\Lambda^2 \Gamma_0 + \Lambda \Gamma'_0 + \gamma_c) - S_{\text{BH}}(\Lambda^2 \Gamma_0 + \Lambda \Gamma'_0 + \gamma_c - \delta) \quad (\text{A.1})$$

$$= \Lambda^4 \left[\frac{\delta^I}{\Lambda^2} \frac{d}{d\Gamma^I} S(\Gamma) \Big|_{\Gamma=\Gamma_0} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right) \right] \quad (\text{A.2})$$

$$= \pi \Lambda^2 \delta^I \frac{d}{d\Gamma^I} |Z(\Gamma, t_*(\Gamma))|^2 \Big|_{\Gamma=\Gamma_0} + \mathcal{O}(1) \quad (\text{A.3})$$

$$= 2\pi \Lambda^2 \text{Re}(\overline{Z}_{\Gamma_0} Z_\delta) \Big|_{t_*(\Gamma_0)} + \mathcal{O}(1) \quad (\text{A.4})$$

$$= \pm 2\pi \Lambda^2 |Z_{\Gamma_0}| m_\delta \Big|_{t_*(\Gamma_0)} + \mathcal{O}(1), \quad (\text{A.5})$$

which indeed diverges when $\Lambda \rightarrow \infty$ except if $m_\delta|_{t_*(\Gamma_0)} = 0$. In going from the first to the second line we used the fact that the Bekenstein-Hawking entropy of a BPS black hole scales quadratically

⁷ The preceding reasoning makes clear why we added the somewhat peculiar term $\Lambda \Gamma'_0$ in (2.4): without it there would be unsuppressed tunneling processes for $\delta \propto \Gamma_0$, and with a term $\Lambda^2 \Gamma'_0$ instead, there would be unsuppressed tunneling for $\delta \propto \Gamma_0 + \Gamma'_0$. Dropping the $\Lambda \Gamma'_0$ term while adding Γ_0 to L_{orb} would be an alternative, but then the awkward situation arises that all walls $W_{k\gamma+m\Gamma_0}$ coincide, spoiling the derivation of the KS formula.

with the charges, and we expanded around $\Lambda = \infty$. In the third line we used the expression of the entropy in terms of the central charge. In going to the next to last line we were allowed to ignore the dependence on Γ through $t_*(\Gamma)$ because $|Z(\Gamma, t)|$ has a critical point at $t = t_*(\Gamma)$, i.e. $\partial_t |Z(\Gamma, t)||_{t_*(\Gamma)} = 0$. In the final step we used the attractor point equation $2 \operatorname{Im}(\overline{Z}_{\Gamma_0} Z_\delta)|_{t_*(\Gamma_0)} = \langle \Gamma_0, \delta \rangle = 0$. Thus, in the absence of massless BPS particles at $t_*(\Gamma_0)$ with charge in L_{orb} , there can be no mixing between galaxies with different core charges.

In conclusion, if no massless charged particles exist at $t_*(\Gamma_0)$, our BPS galaxies are closed quantum systems in the limit $\Lambda \rightarrow \infty$, and the framed index is well defined. Massless charged particles only appear at loci of complex codimension 1. Thus, for a generic Γ_0 , there will be no massless charged particles at $t_*(\Gamma_0)$, and there will be no mixing.

There might however be special circumstances in which we are interested precisely in the situation where L_{orb} contains charges becoming massless at $t_*(\Gamma_0)$. In this case, mixing may occur, so to be guaranteed a well-defined index we should sum over values of the core charge differing by multiples of the charges becoming massless. It is indeed natural to consider such nongeneric situations in compactifications with codimension 1 loci of enhanced gauge symmetry, as we now explain. Near such loci, there are light vector bosons, say of charge γ , and typically also light monopoles of charge γ_D . Their central charges are related by $Z_{\gamma_D} \sim \tau Z_\gamma$, where τ is the (moduli-dependent) complexified coupling, and $Z_\gamma \rightarrow 0$ at the enhanced symmetry locus. When we want to allow both the vector boson and the monopole in a galactic orbit, i.e. $\gamma, \gamma_D \in L_{\text{orb}}$, the attractor equations for Γ_0 imply $\operatorname{Im}(Z_\gamma \overline{Z}_{\Gamma_0}) = 0 = \operatorname{Im}(Z_{\gamma_D} \overline{Z}_{\Gamma_0})$ at $t_*(\Gamma_0)$. Given the relation $Z_{\gamma_D} \sim \tau Z_\gamma$ and $\operatorname{Im} \tau \neq 0$, this implies $Z_\gamma|_{t_*(\Gamma_0)} = 0$ — that is, we necessarily have massless particles at the attractor point. In Section 7 of [1] we discuss an example of this sort, and show explicitly that it is indeed necessary to sum over core charges to get a well-defined index.

To make this more precise, we could try to define a generalized framed index by summing over the entire lattice V_0 of multiples of charges in L_{orb} becoming massless at $t_*(\Gamma_0)$:

$$\overline{\underline{\Omega}}_C(\Gamma_{\text{orb}}; t_\infty) := \sum_{\nu \in V_0} \lim_{\Lambda \rightarrow \infty} \operatorname{Tr}_{\mathcal{H}_{\Gamma_c + \nu}(\Gamma_{\text{orb}} - \nu; t_\infty)} (-1)^F. \quad (\text{A.6})$$

There is some redundancy among these objects, as $\overline{\underline{\Omega}}_{\{\Gamma_0, \Gamma'_0, \gamma_c\}}(\Gamma_{\text{orb}}) = \overline{\underline{\Omega}}_{\{\Gamma_0, \Gamma'_0, \gamma_c + \nu\}}(\Gamma_{\text{orb}} - \nu)$ for any $\nu \in V_0$. Consequently, the associated generating function

$$G_{[C]}(X; t_\infty) := \sum_{\Gamma_{\text{orb}} \in L_{\text{orb}}} \overline{\underline{\Omega}}_C(\Gamma_{\text{orb}}; t_\infty) X^{\gamma_c + \Gamma_{\text{orb}}}, \quad (\text{A.7})$$

depends only on the equivalence class $[C] := \{\Gamma_0, \Gamma'_0, \gamma_c \bmod V_0\}$. We could now try to repeat the analysis of the previous sections using these generalized definitions. It is however not immediately obvious that the objects we have defined here are finite or computable in practice, and indeed it is only in special cases possible to restrict the sum over cores to a finite subset. We will not attempt to provide a general analysis here, but refer to section 7 of [1] for the study of specific examples.

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